An Overview over Models for Fuzzy Quantifiers

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Introduction

Fuzzy quantification is a formal technique that is concerned with quantification over vague concepts and resulting in truth values over $[0, 1]$. Since natural language is pervaded by vagueness and heavily depends on quantificational phenomena fuzzy quantification represents an extremely important aspect in the development of a model of natural language and has been successfully applied to diverse applications such as database querying, data mining and data summarization [5].

According to Glöckner [7] we can distinguish between three main issues in the field of fuzzy quantification, namely, *interpretation*, *summarization* and *reasoning*. The most fundamental of these issues is that of the interpretation of fuzzy quantifiers, which corresponds to the problem of defining the meaning of fuzzy quantifiers i.e. the modeling of vague quantifiers. The problem of reasoning corresponds to the task of inferring new knowledge from possibly fuzzy information by means of fuzzy quantifiers. The issue of summarization describes the problem of aggregating vague data by means of fuzzy quantifiers in order to create meaningful summaries.

The purpose of this paper is to give the reader an introduction to the issue of interpretation of fuzzy quantifiers, which is still not entirely mastered yet and of great importance to the applicability of fuzzy quantification. The paper is organized in several sections. The first section provides the reader with the necessary formal and linguistic background required to be able to understand the issue of finding adequate models to the quantification phenomena occurring in natural language. It discusses the vagueness encountered in natural language, the formal concepts of fuzzy sets, fuzzy logics, and the notion of quantification. The latter sections give an overview over the various approaches proposed for the modeling of natural language quantifiers and discusses these methods as well as their linguistic adequacy. The last section then gives some directions for a deeper study of the field and for further research topics.

1 Background

1.1 Uncertainty of natural language

In order to understand the problem of linguistic adequate modeling of linguistic quantifiers we must understand the vagueness found in natural languages. The uncertainty in natural language arises from the need to express succinctly complex, probably incomplete and ambiguous informations. According to Glöckner [7] the imprecisions of natural language can take the following forms: *vagueness*, *underspecificity*, *ambiguity* and *context dependence*. Even though we will mostly be interested in the issue of handling vagueness we will need to discuss each of these aspects in order to be able to distinguish them properly.

- *Vagueness* is widely agreed to describe concepts that have borderline cases, that cannot be perfectly classified. For example the concepts of baldness, obesity, beauty, or being tall are vague in this sense. Alternatively, concepts are sometimes considered to be vague if they are subject to the *Sorites paradox* [15]. The Sorites paradox is an argument that uses mathematical induction to derive an absurd claim from the assumption that a concept is characterized by a discrete measure. For example, a man can clearly be considered bald if he has no hair. Now, if a bald man has n hairs then he will still be bald if he has one more hair. The iteration of this argument yields that a man with, say, hundred thousand hairs, is bald. This is clearly not the case, which shows that vague concepts are not specifiable in terms of a discrete measure. Furthermore, we can observe that vague concepts are indeed mostly independent from the precision of any measure. Take for example a person that is fifty years old, this person is not clearly not old but also the person is not clearly old. The decision will not become easier if we learn that the person is in effect only forty-nine years, eleven month and twenty days old. This shows that vagueness cannot be resolved by more precise measures. For a more detailed and philosophically motivated discussion about vagueness see [15].

- *Underspecificity* corresponds to a lack of relevant informations, that is usual in communications of incomplete knowledge e.g. "I heard something!" or if information is intentionally retained "Someone told me about the secret.".
- *Ambiguity* refers to multiple distinct meanings of the same word or phrase. For example consider the word "scale" which may among others denote a set of musical notes as well as the rigid plates constituting the skin of reptiles.
- *Context dependence* refers to the variation of the meaning of a concept based on the context. For example a fifteen-year-old person can clearly be considered as young, however, a computer of the same age is evidently old. The basic concept of being old is the same but the meaning varies depending on whether the context is a biological or a technological.

Vagueness is usually modeled by means of many-valued logics such as three-, five- and infinitely-valued logics. In particular fuzzy logic (see section 1.3) fits quite closely to the concept of vagueness, however, it is still questionable if this formalism is really adequate. Indeed, it does not truly capture the notion of borderline cases. The infinitely precise, continuous truth values of fuzzy logic correspond to clear specifications of degrees of belongingness to vague concepts, which is in contrast with the idea of borderline cases that cannot be completely classified.

1.2 Fuzzy Sets

Fuzzy sets are a generalization of classical sets introduced by Zadeh in [21] as a formalism to model the vagueness encountered in the real world. A fuzzy set represents a set, whose elements have a degree of membership in the interval $[0, 1] = \mathbf{I}$. A degree of membership of 0 indicates that an element does not belong to the set, whereas a degree of membership of 1 means that the element fully belongs to the set.

Definition 1 (Fuzzy sets). *Let E be a crisp set, then a fuzzy set A_E over the universe E is characterized by its membership function $\mu_{A_E} : E \to \mathbf{I}$. The support of a fuzzy set X_E is given by* $\text{support}(X_E) = \{e \in E : \mu_{X_E}(e) > 0\}$.

For simplicity we shall discard the subscript of a fuzzy set whenever the choice of the universe is obvious from the context. We will denote concrete fuzzy sets in a similar manner than we do for sets e.g.:

$$X = \{\, 0.33/e_1,\ 0.1/e_2,\ 0.92/e_3 \,\},$$

where the numbers 0.33, 0.1, 0.92 are the degrees of membership of the elements e_1, e_2 and e_3, respectively. To indicate that a fuzzy set models a given concept expressed in natural language we write the name of the fuzzy set in bold letters. For example we write **blond**, **tall**, **bald** to denote the fuzzy sets modeling the concepts of being blond, tall or bald – respectively.

By the previous definition it is easy to see that crisp sets constitute a special case of fuzzy sets, where the range of the characteristic function[1] is restricted to $\{0, 1\} = \mathbf{2}$. The relation $\subseteq$ can be generalized to operate on fuzzy sets. Such a generalization is given by Zadeh in [21] and is defined as follows.

Definition 2 (Fuzzy subset relation)**.** *Let A, B be fuzzy sets over a universe E, then $A \subseteq B :\Longleftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in E$.*

Even though this generalization of $\subseteq$ is not the only possibility, it is seen in the literature as the only adequate choice. In the following we will denote by $\mathcal{P}(E)$ the power set of some crisp set E. Then using the previous definition we can define the fuzzy power set mapping.

Definition 3 (Fuzzy power set)**.** *Let E be a crisp set, then the fuzzy power set of E is defined as $\widetilde{\mathcal{P}}(E) = \{B : B \subseteq E\}$.*

The classical set operations can also be extended to work on fuzzy sets, but unlike the subset relation these operations have many plausible generalizations. The next definition specifies formally the concepts of fuzzy intersection, union and complementation.

Definition 4 (Fuzzy set operations)**.** *Let E be a crisp set. Then mappings $\widetilde{\cap}, \widetilde{\cup} : \widetilde{\mathcal{P}}(E)^2 \to \widetilde{\mathcal{P}}(E)$ and $\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \to \widetilde{\mathcal{P}}(E)$ that satisfy*

$$\widetilde{\cap}|_{\mathcal{P}(E)^2} = \cap\,,\ \ \widetilde{\cup}|_{\mathcal{P}(E)^2} = \cup\,,\ \ \widetilde{\neg}|_{\mathcal{P}(E)} = \neg$$

are respectively called fuzzy intersection, fuzzy union and fuzzy complement[2]*.*

The standard fuzzy set operations, denoted by $\cap, \cup$ and $\neg$, were originally introduced by Zadeh in [21].

Definition 5 (Standard fuzzy set operations)**.** *Let A, B be fuzzy sets over a universe E, then $\cap, \cup$ and $\neg$ are respectively characterized by*

$$\begin{aligned} &\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \\ &\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \\ &\mu_{\neg A}(x) = 1 - \mu_A(x),\ \textit{for all } x \in E. \end{aligned}$$

[1]We denote the characteristic function of a crisp set by χ.

[2]The notation $f|_C$ denotes the restriction of the domain of the function $f : A \to B$ to $C \subseteq A$.

It is easy to see that $\cap$, $\cup$, and $\neg$ satisfy the conditions formulated in definition 4 and are, therefore, generalizations of the classical set operations.

Occasionally we will need to systematically extend functions operating on given base sets to operate on fuzzy sets over the base set. This leads to the definition of so-called extension principles which roughly correspond to fuzzification mechanisms for functions.

Definition 6 (Extension principle)**.** *An extension principle $\mathcal{E}$ assigns to each $f : E \to E'$ a mapping $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \to \widetilde{\mathcal{P}}(E')$, where $E, E' \neq \emptyset$.*

We also introduce the related notion of powerset mapping which describes the extension of a function to the powerset of its domain. The explicit distinction between functions and their powerset mapping will later allow us to avoid ambiguities.

Definition 7 (Powerset mapping)**.** *Let $f : E \to E'$ then we define the powerset mapping $\hat{f} : \mathcal{P}(E) \to \mathcal{P}(E')$ by $\hat{f}(X) = \{f(e) : e \in X\}$ for all $X \in \mathcal{P}(E)$.*

Given a fuzzy set, we are sometimes interested in the set of elements that satisfy at least a certain degree of membership – this corresponds to the notion of α-cut.

Definition 8 (α-cut)**.** *Let $A \in \widetilde{\mathcal{P}}(E)$ and let $\alpha \in \mathbf{I}$, then the α-cut of A is the crisp set $A_{\geq\alpha} = \{x \in E : \mu_A(x) \geq \alpha\}$. The strict α-cut of A is given by $A_{>} = \{x \in E : \mu_A(x) > \alpha\}$.*

Later we will need to retrieve in descending order the degrees of membership occurring in a given fuzzy set. For this purpose we introduce an additional notation.

Definition 9. *Let $E \neq \emptyset$ be a finite set with $|E| = m$ and $X \in \widetilde{\mathcal{P}}(E)$. Furthermore let $E = \{e_1, \ldots, e_m\}$ such that $\mu_X(e_1) \geq \ldots \mu_X(e_m)$. Then we define*

$$\mu_{[j]}(X) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j > m \\ \mu_X(e_j) & \text{otherwise.} \end{cases}$$

1.3 Fuzzy Logic

Fuzzy logic represents a family of logics whose development was motivated by the introduction of fuzzy sets. Fuzzy logic extends classical bi-valued logics by allowing continuous truth values in the interval $\mathbf{I}$, where 0 represents falsity and 1 represents absolute truth. The connectives $\widetilde{\neg}$, $\widetilde{\vee}$, $\widetilde{\wedge}$ and the quantifiers $\widetilde{\exists}$, $\widetilde{\forall}$ of fuzzy logic, unlike their classical counterparts, do not have a fixed meaning. The only condition they have to satisfy is to correctly generalize the classical analogues, i.e. $\widetilde{\neg}|_{\mathbf{2}} = \neg, \widetilde{\vee}|_{\mathbf{2}} = \vee$ and $\widetilde{\wedge}|_{\mathbf{2}} = \wedge$.

Definition 10 (Standard fuzzy connectives)**.** *We denote by $\neg : \mathbf{I} \to \mathbf{I}, \vee : \mathbf{I}^2 \to \mathbf{I}$ and $\wedge : \mathbf{I}^2 \to \mathbf{I}$ the standard fuzzy connectives given by*

$$\begin{aligned} \neg x &= 1 - x \\ x \vee y &= \max\{x, y\} \\ x \wedge y &= \min\{x, y\}, \text{ for all } x, y \in \mathbf{I}. \end{aligned}$$

The most commonly used family of fuzzy logics are t-norm fuzzy logics [9], which use t-norms as the meaning of the $\widetilde{\wedge}$ connective and t-conorms as the meaning of $\widetilde{\vee}$.

Definition 11 (t-norm and t-conorm)**.** *A t-norm is a function* $\top : \mathbf{I}^2 \to \mathbf{I}$ *that satisfies the following properties.*

$$\begin{aligned} &\top(a,b) = \top(b,a) && \textit{(Commutativity)}\\ &\top(a,b) \leq \top(c,d) \quad \textit{if } a \leq c \textit{ and } b \leq d && \textit{(Monotonicity)}\\ &\top(a,\top(b,c)) = \top(\top(a,b),c) && \textit{(Associativity)}\\ &\top(a,1) = 1 && \textit{(Neutral Element 1).} \end{aligned}$$

The corresponding t-conorm $\bot$ *is defined as* $\bot(a,b) = 1 - \top(1-a, 1-b)$.

In fuzzy predicate logic predicates are usually interpreted as fuzzy sets. Then in case of continuous t-norm fuzzy logics, the sentences $\widetilde{\forall} x \varphi(x)$ and $\widetilde{\exists} x \varphi(x)$ are respectively interpreted as the infimum and the supremum of the truth values of $\varphi(x)$.

1.4 Quantification

1.4.1 The concept of quantification

From an abstract point of view the concept of quantification allows us to express properties of a collection of individuals. It corresponds thus to a second order concept. In natural language quantification is ubiquitous. By means of names, articles, temporal adverbs, spacial adverbs it can refer – more or less explicitly – to quantities, individuals, points in time, points in space, situations, etc. Some examples of quantification are:

- "*John* is here.", "*The* man brought flowers."
- "*Most* people like each other.", "*Much* wine was drunk."
- "The champion *always* won."
- "The key is *somewhere*."

Besides being ubiquitous in natural language, quantification also has a great impact on the meaning of sentences e.g. quantification makes the difference between "having no money" and "having a lot of money" [7].

Quantification is known to logic since its early beginnings in the antiquity. However the formal logic restricted its attention mostly to the formal counterparts $\forall$ and $\exists$ of the quantifiers "everything" and "something" that are commonly interpreted as unary quantifiers over a base set. It was only in 1957 that the logician Mostowski proved that some quantifiers are not expressible in terms of $\forall$ and $\exists$ [12]. This led to the creation of the theory of generalized quantifiers.

1.4.2 Generalized Quantifiers

The concept of generalized quantifiers arose from the discovery of many mathematical and linguistic quantifiers, which are not expressible in predicate logic restricted to $\forall$ and $\exists$. Generalized quantifiers are defined to be mappings from subsets or relations over a base set E to a binary truth value. The generalized quantifiers were analyzed by Barwise and Cooper for their relationship to natural languages who found that they were able to express numerous linguistic quantifiers. Moreover, Barwise and Cooper embedded these quantifiers in a logical language that reflects the syntax of natural language much better than the traditional predicate logic does [1].

1.4.3 Fuzzy Quantifiers

The ability of generalized quantifiers to handle natural language concepts as well as the introduction of fuzzy sets by Zadeh and the subsequent development of fuzzy logic motivated the extension of generalized quantifiers to fuzzy quantifiers in order to handle inherently vague quantifiers of natural language like “few”, “many”, “approximately half of”, “much more than”, etc. We can think of fuzzy quantifiers as mappings from fuzzy subsets of a base set to a fuzzy truth value.

1.4.4 Terminology

In the following we will define some of the terminology related to quantifiers in order to avoid ambiguities between the mathematical concepts and the linguistic concepts. First we need to formally define the notion of quantifier, which we restrict to the case of monadic[3] crisp quantifiers.

Definition 12 (Quantifiers)**.** *A (crisp) quantifier on a base set* $E \neq \emptyset$ *is a mapping* $Q : \mathcal{P}(E)^n \to \mathbf{2}$

Because of this restriction our notion of quantifier will not capture entirely the above, informally stated notion of generalized quantifiers, this is however acceptable since we will mostly be concerned by monadic quantification. Similarly, we define the notion of semi-fuzzy quantifiers and fuzzy quantifiers.

Definition 13 (Semi-fuzzy quantifiers)**.** *A semi-fuzzy quantifier on a base set* $E \neq \emptyset$ *is a mapping* $Q : \mathcal{P}(E)^n \to \mathbf{I}$.

Definition 14 (Fuzzy quantifiers)**.** *A fuzzy quantifier on a base set* $E \neq \emptyset$ *is a mapping* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$.

In a mathematical context we will sometimes have to refer to quantifiers of natural language. To this end we use the notion of *linguistic quantifiers*, which refers to quantifier symbols of natural language and their intuitive meaning. Just as in the case of fuzzy sets we use bold letters to denote quantifiers modeling a linguistic quantifier. For instance we denote by **most**, **exactly one**, **only** the quantifiers respectively modeling the linguistic quantifiers “most”, “exactly one” and “only”.

1.5 Classes of Quantifiers

Because of the wealth of quantifiers occurring in natural language and in mathematics it was necessary to establish classifications that group quantifiers by their structural and semantic characteristics. In the following we will consider

[3] We say that a quantifier is monadic if all of its arguments are monadic (see section 1.5). The adjective “unary” is used to express that a quantifier has only one argument.

two important classifications: one for linguistic quantifiers and one for logical quantifiers.

1.5.1 Absolute and Relative Quantifiers

The classes of absolute and relative quantifiers are subclasses of unary and binary quantitative linguistic quantifiers. Absolute quantifiers represent absolute quantities and intervals, whereas relative quantifiers express proportions and percentages. Some examples of absolute quantifiers are “exactly one”, “approximately five”, “exists”. The quantifiers “at least half”, “around one third” are examples of relative quantifiers.

These classes will be particularly relevant in the first part of the overview where we will see the methods based on fuzzy linguistic quantifiers, which mainly concentrate on the modeling of absolute and relative quantifiers.

Even though these two classes of linguistic quantifiers are of great practical interest, many other classes of practically relevant linguistic quantifiers such as *quantifiers of exception*, *cardinal comparatives*, *proportional comparatives* etc. exist [5, 7].

1.5.2 Monadic and Polyadic Quantifiers

The notions of monadic and polyadic quantifiers are of importance for the understanding of the scope of the methods considered in the sections 2, 3 and 4. Indeed, they allow us to draw a line between the usual concept of quantifiers in the context of fuzzy quantification and the concept of generalized quantifiers. Let us begin by introducing the closely related concept of types, which will allow us to define quantifiers of arbitrary types.

Definition 15 (Types)**.** *A type is a tuple of the form $\langle t_1, \ldots, t_n \rangle$ where $n \in \mathbb{N}^+$ and $t_i \in \mathbb{N}^+$ for $i = 1, \ldots, n$.*

A quantifier of type $\langle t_1, \ldots, t_n \rangle$ is a mapping of the form $Q : \mathcal{P}(E^{t_1}) \times \cdots \times \mathcal{P}(E^{t_n}) \to \mathbf{2}$ – analogously we define semi-fuzzy and fuzzy quantifiers of type $\langle t_1, \ldots, t_n \rangle$. Quantifiers of type $\langle 1, \ldots, 1 \rangle$ are said to be *monadic*, while quantifiers of any other type are said to be *polyadic*[4]. Thus, our notion of quantifiers (see definitions 12, 13, 14) covers the monadic case only. This is because almost all approaches to fuzzy quantification can only generate monadic models and are thus not suited to model more complex but relevant constructs such as branching quantification (see section 4).

1.6 Structure of the approaches to fuzzy quantification

In the following we will briefly discuss the general structure of approaches to fuzzy quantification. Indeed all current approaches follow a two-step scheme which consists in the specification of fuzzy quantifiers using some suitable medium and the systematic translation of the specification to a fuzzy quantifier. The specification step is of course application dependent and has, therefore, to be carried out manually. It is thus the translation step step which constitutes the core of any approach to fuzzy quantification. While inspecting the individual

[4]This terminology in analogy to the terminology used by Westerståhl for generalized quantifiers in [16]

approaches we will, hence, focus on the description and the analysis of the translation procedures. Formally, we can represent such a translation procedure as a mapping

$$\mathscr{F} : \mathscr{S} \to \mathscr{Q},$$

where $\mathscr{S}$ represents the space of specifications and $\mathscr{Q}$ is the space of fuzzy quantifiers. Depending on the nature of the specification medium we can distinguish between two main categories of methods, namely the family of approaches based on fuzzy linguistic quantifiers (see section 2) and the family of approaches using semi-fuzzy quantifiers (see section 3).

1.7 Adequacy

Since we are modeling linguistic quantifiers we need some criteria to determine whether our models are adequate or not i.e. we need to check whether a fuzzy quantifier that models some linguistic quantifier reflects its intuitive meaning and its properties. This is commonly referred to as *linguistic adequacy.* This notion is rather informal and general, which means that we cannot verify whether some models are truly adequate. Instead we can formulate adequacy constraints which, if they hold, give us evidence about the linguistic adequacy. Additionally, we can specify concrete examples in which a model exhibits implausible behavior.

There are mainly two ways to specify adequacy criteria for fuzzy models of linguistic quantifiers. The first method is to specify explicitly which properties are to be satisfied by a model of a given linguistic quantifier. The second way to specify adequacy constraints is to require the preservation of a given property during the translation process from the specification medium to the resulting fuzzy quantifier. This has the advantage of being much more systematic and allows us to obtain more general results about the adequacy of the considered method. The adequacy requirements for the methods of the fuzzy linguistic quantifier family are usually specified in the former way because the incompatible structure of fuzzy linguistic quantifiers and fuzzy quantifiers makes the specification of the preservation of properties rather difficult. The methods based on semi-fuzzy quantifiers – especially Glöckner's method – specify adequacy requirements in the latter way.

In the following we will distinguish between semantic properties and quantifier constructions, which are two important instruments for the specification of linguistic adequacy.

1.7.1 Semantic properties

Semantic properties describe the behavior of quantifiers. The properties described in the following are inspired by properties found in natural language quantifiers. Usually, we require that adequate models exhibit semantical properties that reflect the intuitive behavior as closely as possible.

The first property we will consider is that of monotonicity in an argument, which specifies that the truth value of a quantified sentence increases/decreases if an argument becomes more/less general. To clarify this consider the sentence "Some men are tall", which must be at least as true as some "Some blond men are tall" because "some" has a monotonic increasing behavior in its restriction[5] and the concept of men is more general than that of blond men.

Definition 16 (Monotonicity in the i-th argument). *A fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ *is said to be increasing in its i-th argument* $i \in \{1, \ldots, n\}$*, if*

$$\widetilde{Q}(X_1, \ldots, X_n) \leq \widetilde{Q}(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n),$$

whenever $X_1, \ldots, X_n, X_i' \in \widetilde{\mathcal{P}}(E)$ *satisfy* $X_i \subseteq X_i'$*. Q is said to be decreasing in its i-th argument, if under the same conditions it always holds that*

$$\widetilde{Q}(X_1, \ldots, X_n) \geq \widetilde{Q}(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n).$$

The corresponding definitions for semi-fuzzy quantifiers are analogous with the arguments ranging over $\mathcal{P}(E)$ *and* $\subseteq |_{\mathcal{P}(E)^2}$*.*

A few more examples of monotonic linguistic quantifiers are: "all", "no", "some", "many", "at least three".

Another important property is that of *"having extension"*, which is given if a quantifier is insensitive to the addition of irrelevant elements to the base set. For example the sentence "Some men are tall" should have the same value on the base set consisting of all men as well as on any other base set containing all men.

Definition 17 (Having extension). *Let* $E \subseteq E'$ *be base sets and let* $Q_E : \mathcal{P}(E)^n \to \mathbf{I}$ *and* $Q_{E'} : \mathcal{P}(E')^n \to \mathbf{I}$ *be the interpretation of Q over E and E', respectively. Then Q is said to have extension if*

$$Q_E(X_1, \ldots, X_n) = Q_{E'}(X_1, \ldots, X_n), \text{ for all } X_1, \ldots, X_n \in \mathcal{P}(E).$$

In an analogous way we define the notion of having extension for fuzzy quantifiers.

The property of *conservativity* expresses another form of context insensitivity that is pervasive in natural language quantifiers. A (semi-) fuzzy quantifier is conservative if the quantification is unaffected by elements in the scope that do not belong to the restriction.

Definition 18 (Conservativity). *A semi-fuzzy quantifier* $Q : \mathcal{P}(E)^2 \to \mathbf{I}$ *is conservative if*

$$Q(X_1, X_2) = Q(X_1, X_1 \cap X_2),$$

for all $X_1, X_2 \in \mathcal{P}(E)$*.*

For fuzzy quantifiers we can distinguish between two degrees of conservativity, namely, weak and strong conservativity.

Definition 19 (Weak and strong conservativity). *A fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^2 \to \mathbf{I}$ *is weakly conservative if*

$$\widetilde{Q}(X_1, X_2) = \widetilde{Q}(X_1, \text{support}(X_1) \cap X_2), \quad \text{for all } X_1, X_2 \in \widetilde{\mathcal{P}}(E).$$

[5] In the case of two-place quantification of the form "Q of X_1 are X_2", the first argument, X_1, is called the quantifiers *restriction*, since it restricts the quantification to the individuals in X_1. The second argument, X_2, which expresses some property of the individuals is called the *scope*.

Furthermore, given a fuzzy intersection $\widetilde{\cap}$, $\widetilde{Q}$ *is said to be strongly conservative if*

$$\widetilde{Q}(X_1, X_2) = \widetilde{Q}(X_1, X_1 \widetilde{\cap} X_2), \quad \text{for all } X_1, X_2 \in \widetilde{\mathcal{P}}(E).$$

The preservation of conservativity in the strong sense from semi-fuzzy quantifiers turns out to be incompatible with the preservation of other useful properties. This is the reason for the introduction of weak conservativity, which still expresses the aspect of context insensitivity while not excluding the other properties [7].

To illustrate concept of conservativity consider the sentence "Most of X are Y". This sentence can be equivalently reformulated as "Most of X are X and Y". Hence, the quantifier "most" is intuitively conservative. A few other examples of conservative quantifiers are "all", "exists", "few", "many". An example of a non-conservative quantifier is the quantifier "only". For instance consider the sentence "Only blond people are tall", which is not equivalent to the tautological sentence "Only blond people are blond and tall people".

The last semantic property that we will mention here is that of automorphism invariance, which formalizes the concept of quantitative quantifiers. A quantifier is said to be quantitative if it expresses a quantity and is, thus, insensitive to specific individuals of the base set. A few examples of quantitative quantifiers are "all", "exists", "many". Non-quantitative quantifiers are also called *qualitative.* In natural language proper names are examples of non-quantitative quantifiers.

Definition 20 (Quantitativity)**.** *A semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *is quantitative if for all automorphisms* $\xi : E \to E$ *and all* $X_1, \ldots, X_n \in \mathcal{P}(E)$

$$Q(X_1, \ldots, X_n) = Q(\widehat{\xi}(X_1), \ldots, \widehat{\xi}(X_n)).$$

A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ *is quantitative w.r.t. to an extension principle* $\mathcal{E}$ *if for all automorphisms* $\xi : E \to E$ *and all* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$

$$\widetilde{Q}(X_1, \ldots, X_n) = \widetilde{Q}(\mathcal{E}(\xi)(X_1), \ldots, \mathcal{E}(\xi)(X_n)).$$

Of course the list of properties discussed above is far from being complete, there exist many others like e.g. *convexity*, *propagation of fuzziness*, *cylindrical extension*, etc. More extensive collections of properties can be found in [5, 7]

1.7.2 Quantifier constructions

Quantifier constructions, as the name suggests, allow us to construct new quantifiers from existing ones, thereby, implicitly relating the quantifiers. In the following we will consider some quantifier constructions that model very similar constructions found in natural language.

Antonyms and duals are two important notions relating natural language quantifiers. The notion of antonym describes quantifiers that have an opposite meaning such as "all" and "no", "at least half" and "at most half". We model antonyms by the complementation of the last argument.

Definition 21 (Antonyms)**.** *The antonym* $Q\neg : \mathcal{P}(E)^n \to \mathbf{I}$ *of a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *with* $n > 0$ *is defined by*

$$Q\neg(X_1, \ldots, X_n) = Q(X_1, \ldots, \neg X_n)$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$*. The antonym* $\widetilde{Q}\widetilde{\neg} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ *of a fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ *is defined analogously, based on the fuzzy complement* $\widetilde{\neg}$*.*

The notion of dual corresponds to the negation of the antonym. Some examples of linguistic quantifiers and their duals are: "all" and "some", "many" and "few", "at most 70%" and "less than 30%".

Definition 22 (Duals)**.** *Given a fuzzy negation* $\widetilde{\neg}$*, the dual* $\widetilde{\neg}Q\neg : \mathcal{P}(E)^n \to \mathbf{I}$ *of a semi-fuzzy quantifier* Q *is defined by*

$$\widetilde{\neg}Q\neg(X_1, \ldots, X_n) = \widetilde{\neg}(Q\neg(X_1, \ldots, X_n))$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$*. In a completely analog way we define the dual* $\widetilde{\neg}\widetilde{Q}\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$ *of a fuzzy quantifier* $\widetilde{Q}$*.*

If the fuzzy negation is fixed by the context we denote the dual of a semi-fuzzy quantifier Q by $Q\widetilde{\square}$ — and analogously for fuzzy quantifiers.

Another construction that comes to mind is that of unions and intersections of arguments to model sentences that have a combination in some of their arguments e.g. "Most athletes are young and tall".

Definition 23 (Unions and intersections of arguments)**.** *Let* $E \neq \emptyset$ *and* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *be a semi-fuzzy quantifier with* $n > 0$*. Then, the semi-fuzzy quantifier* $Q\cup : \mathcal{P}(E)^n \to \mathbf{I}$ *is given by*

$$Q\cup(X_1, \ldots, X_n, X_{n+1}) = Q(X_1, \ldots, X_n \cup X_{n+1})$$

for all $X_1, \ldots, X_n, X_{n+1} \in \mathcal{P}(E)$*. Analogously we define the fuzzy quantifier* $\widetilde{Q}\widetilde{\cup}$ *based on the fuzzy union* $\widetilde{\cup}$*. Similarly we define the semi-fuzzy quantifier and the fuzzy quantifier* $Q\cap$ *and* $\widetilde{Q}\widetilde{\cap}$*.*

Consider again the example above. If the linguistic quantifier most is modeled by $\mathbf{most} : \mathcal{P}(E)^2 \to \mathbf{I}$, then the sentence above could be modeled by $\mathbf{most} \cap (\mathbf{athlete}, \mathbf{young}, \mathbf{tall})$. So far, boolean operations on arguments are restricted to the last argument of a quantifier. In natural language, however, such constructions on arguments may occur in any argument of a quantified statement. To handle these cases we introduce the notion of argument permutation.

Definition 24 (Argument permutation)**.** *Let* $E \neq \emptyset$*,* $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *be a semi-fuzzy quantifier and* $\beta : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *be a permutation. Then, the semi-fuzzy quantifier* $Q\beta : \mathcal{P}(E)^n \to \mathbf{I}$ *is given by*

$$Q\beta(X_1, \ldots, X_n) = Q(X_{\beta(1)}, \ldots, X_{\beta(n)})$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$*. Given a fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \to \mathbf{I}$ *we define* $\widetilde{Q}\beta$ *analogously.*

This construction also occurs naturally in the quantifiers "only" and "all". Indeed the sentences "Only X_1 are X_2" and "All X_2 are X_1" are equivalent. Hence, we can model the linguistic quantifier "only" by $\mathbf{only}(X_1, X_2) = \mathbf{all}\beta(X_1, X_2)$, where $\beta = \{1 \mapsto 2, 2 \mapsto 1\}$.

Again, this list of quantifier constructions is non-exhaustive. Many other interesting constructions such as *quantifier conjunctions*, *multiple occurrences of variables* etc. can be defined [7].

2 Methods based on fuzzy linguistic quantifiers

This section covers some, but not all of the methods based on fuzzy linguistic quantifiers. A more complete collection of methods can be found in [5]. The approaches based on fuzzy linguistic quantifiers are, compared to the ones based on semi-fuzzy quantifiers, rather simple and are mostly limited to the cases of one- and two-place quantification over finite base sets. On the other hand they are still practically relevant because of their easier implementation and their lower computational complexity.

Before we begin with the examination of the individual approaches we need to formally define the notion of fuzzy linguistic quantifiers and some related concepts. A fuzzy linguistic quantifier associates truth values to scalar cardinalities i.e. they can also be considered as fuzzy subsets of the reals, for this reason we will also use the symbol μ to denote fuzzy linguistic quantifiers.

Definition 25 (Fuzzy linguistic quantifier)**.** *Let Q be a linguistic quantifier, then a mapping $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ or $\mu_Q : \mathbf{I} \to \mathbf{I}$ modeling the meaning of Q is called a fuzzy linguistic quantifier.*

The actual type of a fuzzy linguistic quantifier μ_Q depends on whether the linguistic quantifier Q is to be modeled as an absolute quantifier, in which case $\mu_Q \in \mathbf{I}^{\mathbb{R}^+}$, or as a relative quantifier in which case $\mu_Q \in \mathbf{I}^{\mathbf{I}}$. Sometimes we need to generate an absolute fuzzy linguistic quantifier from a relative one.

Definition 26. *Let $\mu_Q : \mathbf{I} \to \mathbf{I}$ and E be a finite set, then the mapping $\mu_{Q,E} : \{0, \dots, |E|\} \to \mathbf{I}$ is defined as $\mu_{Q,E}(i) = \mu_Q(\frac{i}{|E|})$ for all $i \in \{0, \dots, |E|\}$.*

While considering the different approaches we will sometimes have to distinguish fuzzy linguistic quantifiers depending on whether they are increasing, decreasing and whether they are *regular*. These concepts are formally described in the next definition.

Definition 27. *A fuzzy linguistic quantifier $\mu_Q : \mathcal{D} \to \mathbf{I}$ with $\mathcal{D} \in \{\mathbb{R}^+, \mathbf{I}\}$ is increasing (decreasing) if it satisfies $\mu_Q(x_1) \leq \mu_Q(x_2)$ ($\mu_Q(x_1) \geq \mu_Q(x_2)$) for all $x_1, x_2 \in \mathcal{D}$ with $x_1 \leq x_2$.*
Furthermore, if an increasing (decreasing) $\mu'_Q : \mathbf{I} \to \mathbf{I}$ satisfies $\mu'_Q(0) = 0$ and $\mu'_Q(1) = 1$ ($\mu'_Q(0) = 1$ and $\mu'_Q(1) = 0$) then it is called regular increasing or coherent (regular decreasing). We call an increasing (decreasing) $\mu'_Q : \mathbb{R}^+ \to \mathbf{I}$ regular increasing (regular decreasing) if it satisfies $\mu'_Q(0) = 0$ and $\mu'_Q(|E|) = 1$ ($\mu'_Q(0) = 1$ and $\mu'_Q(|E|) = 1$ where E is a finite base set.

We can now describe the general form of an evaluation method based on fuzzy linguistic quantifiers. These approaches construct fuzzy quantifiers from fuzzy linguistic quantifiers given as the specification, thus, we can consider these models as functionals that map fuzzy linguistic quantifiers to fuzzy quantifiers. As it was already mentioned earlier the approaches are able to handle the cases of absolute and relative one- and two-place quantification only. This means each method can be seen as a collection of at most four functionals covering the respective cases of quantification. This is more formally expressed in the next definition.

Definition 28 (Models based on fuzzy linguistic quantifiers)**.** *An evaluation method $\mathcal{Z}$ based on fuzzy linguistic quantifiers consists of at least one of the* *following four functionals*

$$\begin{aligned}\mathcal{Z}^{(1)}_{\text{abs}} &: (\mathbb{R}^+ \to \mathbf{I}) \to (\widetilde{\mathcal{P}}(E) \to \mathbf{I}),\\ \mathcal{Z}^{(2)}_{\text{abs}} &: (\mathbb{R}^+ \to \mathbf{I}) \to (\widetilde{\mathcal{P}}(E)^2 \to \mathbf{I}),\\ \mathcal{Z}^{(1)}_{\text{rel}} &: (\mathbf{I} \to \mathbf{I}) \to (\widetilde{\mathcal{P}}(E) \to \mathbf{I}),\\ \mathcal{Z}^{(2)}_{\text{rel}} &: (\mathbf{I} \to \mathbf{I}) \to (\widetilde{\mathcal{P}}(E)^2 \to \mathbf{I}),\end{aligned}$$

which model – as indicated by the sub- and superscripts – the absolute unary, absolute binary, relative unary and relative binary cases of quantification, respectively.

In the literature [5, 7] we find three more or less explicit requirements that an evaluation method $\mathcal{Z}$ should satisfy. First it is required that

$$\mathcal{Z}^{(2)}_{\text{abs}}(\mu_Q)(X_1, X_2) = \mathcal{Z}^{(1)}_{\text{abs}}(\mu_Q)(X_1 \,\widetilde{\cap}\, X_2),$$

i.e. that sentences like “exactly three men are tall” are equivalent to “exactly three things are men and tall”.

Furthermore, there are two common assumptions that specify the behavior in the case of unary relative quantification. It is often assumed that

$$\mathcal{Z}^{(1)}_{\text{rel}}(\mu_Q) = \mathcal{Z}^{(1)}_{\text{abs}}(\mu_{Q,E}),$$

which is only possible on finite base sets where proportions are quantitative. Finally, it is quite commonly required that an unrestricted relative fuzzy quantifier behaves as the corresponding binary relative quantifier with the base set as restriction, i.e.

$$\mathcal{Z}^{(1)}_{\text{rel}}(\mu_Q)(X_1) = \mathcal{Z}^{(2)}_{\text{rel}}(\mu_Q)(E, X_1).$$

This means a sentence like e.g. “Half of things are expensive” is considered to be equivalent to “Half of the things, that are things are expensive”.

2.1 Sigma-count approach

The sigma-count approach was first introduced by Zadeh in [20]. This approach is based on the sigma-count cardinality of fuzzy sets and is meant to model absolute and relative, one- and two-place quantifiers.

Definition 29 (Sigma-count)**.** *Let E be a finite set then $\Sigma_{\text{count}} : \widetilde{\mathcal{P}}(E) \to \mathbb{R}^+$ is defined as*

$$\Sigma_{\text{count}}(A) = \sum_{x \in E} \mu_A(x), \text{ for all } A \in \widetilde{\mathcal{P}}(E).$$

The relative sigma-count $\Sigma_{\text{count}} : \widetilde{\mathcal{P}}(E)^2 \to \mathbf{I}$ is defined as

$$\Sigma_{\text{count}}(B/A) = \frac{\Sigma_{\text{count}}(A \cap B)}{\Sigma_{\text{count}}(A)}, \text{ for all } B \in \widetilde{\mathcal{P}}(E) \text{ and all } A \in \widetilde{\mathcal{P}}(E) \setminus \{\}.$$

The case $\Sigma_{\text{count}}(B/\{\})$ has been silently ignored in the literature, therefore we will not give a complete definition. The fuzzy quantifiers expressed using the relative sigma-count will, thus, be undefined in the case of the restriction being the empty set.

Definition 30 (Sigma-count models)**.** *Let E be a finite set, then the sigma-count models are generated by the following four functionals.*

$$\mathrm{SC}^{(1)}_{\mathrm{abs}}(\mu_Q)(X_1) = \mu_Q(\Sigma_{\mathrm{count}}(X_1))$$
$$\mathrm{SC}^{(2)}_{\mathrm{abs}}(\mu_Q)(X_1, X_2) = \mu_Q(\Sigma_{\mathrm{count}}(X_1 \cap X_2))$$

for all $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$;

$$\mathrm{SC}^{(1)}_{\mathrm{rel}}(\mu_Q)(X_1) = \mathrm{SC}^{(2)}_{\mathrm{rel}}(E, X_1)$$
$$\mathrm{SC}^{(2)}_{\mathrm{rel}}(\mu_Q)(X_1, X_2) = \mu_Q(\Sigma_{\mathrm{count}}(X_2/X_1))$$

for all $\mu_Q : \mathbf{I} \to \mathbf{I}, X_1 \in \widetilde{\mathcal{P}}(E) \setminus \{\}$ and $X_2 \in \widetilde{\mathcal{P}}(E)$.

The sigma-count approach has several adequacy limitations. The first one that we will examine here is due to the fact that the sigma-count aggregates the membership degrees of fuzzy sets in such a way that one large membership degree becomes indistinguishable from many smaller degrees. This problem is depicted in the following example.

Example 1 ([7]). We want to model the linguistic quantifier **exactly one** by means of the fuzzy linguistic quantifier

$$\mu_{\mathbf{exactly\,one}}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the set **child** $= \{\mathrm{max}, \mathrm{susi}\}$ and the fuzzy set **hungry** $= \{0.5/\mathrm{max}, 0.5/\mathrm{susi}\}$. The evaluation of the sentence "exactly one child is hungry" using the sigma-count models leads to the following result

$$\begin{aligned}\mathrm{SC}^{(1)}_{\mathrm{abs}}(\mu_{\mathbf{exactly\,one}})(\mathbf{child}, \mathbf{hungry}) &= \mu_{\mathbf{exactly\,one}}(\Sigma_{\mathrm{count}}(\mathbf{hungry}/\mathbf{child})) \\ &= \mu_{\mathbf{exactly\,one}}(\Sigma_{\mathrm{count}}(\{0.5/\mathrm{max}, 0.5/\mathrm{susi}\})) \\ &= 1.\end{aligned}$$

□

The previously obtained result is clearly inadequate and it is easy to see that this behavior is not due to a wrong choice of the fuzzy linguistic quantifier $\mu_{\mathbf{exactly\,one}}$. Indeed, the only way to produce another output would be to require $\mu_{\mathbf{exactly\,one}}(1) \neq 1$ but this would again yield an inadequate result in the case of, say, **hungry** $= \{1/\mathrm{max}, 0/\mathrm{susi}\}$.

Another problem with the sigma-count approach is its discontinuity in presence of two-valued fuzzy linguistic quantifiers. In this case it is obvious by definition 30 that the sigma-count models produce a two valued output. As shown in the next example, this can result in opposite truth values for inputs that are very similar.

Example 2 (Sigma-count discontinuity). Consider again the set **child** and the fuzzy linguistic quantifier $\mu_{\mathbf{exactly\,one}}$ from the previous example. Then we would expect the evaluation for **hungry** $= \{0.99/\mathrm{max}, 0/\mathrm{susi}\}$ to yield a value quite close to 1, however the sigma-count model produces the following:

$$\mathrm{SC}^{(1)}_{\mathrm{abs}}(\mu_{\mathbf{exactly\,one}})(\mathbf{child}, \mathbf{hungry}) = 0$$

□

This is of great practical relevance since data very often is a little noisy and can thus produce very different results. More examples of this inadequate behavior can be found in [7]. Again this behavior does not depend on the choice of the fuzzy linguistic quantifier as it can be shown that there is not always an adequate continuous valued fuzzy linguistic quantifier [7].

2.2 OWA approach

The OWA approach, initially presented by Yager in [18], is unlike the sigma-count not based on a cardinality count, instead it uses so-called OWA[6] operators to realize the aggregation. The OWA approach is intended to generate models from regular increasing fuzzy linguistic quantifiers.

Definition 31 (OWA operator)**.** *An n-dimensional OWA operator with $n \geq 1$, is a mapping $O_{\mathbf{w}} : \mathbf{I}^n \to \mathbf{I}$ defined by*

$$O_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{w}_i \mathbf{x}_{[i]} \quad \textit{for all } \mathbf{x} \in \mathbf{I}^n,$$

where $\mathbf{w} \in \mathbf{I}^n$, $\sum_{i=1}^{n} \mathbf{w}_i = 1$ and $\mathbf{x}_{[j]}$ with $1 \leq j \leq n$ denotes the j-th largest element of $\mathbf{x}$.

According to Yager [18] an OWA operator has a behavior that lies between the "and" and "or" aggregations. The so-called *orness* of an OWA operator is a value in $\mathbf{I}$ that describes to which degree the operator behaves as an "or" aggregation. This measure is required in the formalization of the OWA models for two-place quantification.

Definition 32 (Orness of an OWA operator)**.** *Let $O_{\mathbf{w}}$ be an n-dimensional OWA operator, then its orness is given by:*

$$\text{orness}(\mathbf{w}) = \frac{1}{n-1} \sum_{j=1}^{n} (n-j) \cdot \mathbf{w}_j.$$

In particular if E is a finite set with $|E| > 1$ we define for any coherent $\mu_Q : \mathbf{I} \to \mathbf{I}$

$$\text{orness}(\mu_{Q,E}) = \frac{1}{n-1} \sum_{j=1}^{n} (n-j) \cdot (\mu_{Q,E}(j) - \mu_{Q,E}(j-1)).$$

It is important to notice that the orness is undefined if the base set E contains only a single element. We can furthermore express this measure more succinctly as stated in the next proposition.

Proposition 1. *Let E be a finite set with $|E| > 1$, then for any coherent $\mu_Q : \mathbf{I} \to \mathbf{I}$ the following holds:*

$$\text{orness}(\mu_{Q,E}) = \frac{1}{n-1} \sum_{j=1}^{n-1} \mu_{Q,E}(j).$$

[6] The acronym OWA stands for "ordered weighted averaging".

This proposition can be proven quite easily by decomposing the sum in the expression of the orness. Having defined the measure of orness for coherent fuzzy linguistic quantifiers, we are ready to introduce the reduction function Δ_q that is required by the OWA method to model two-place quantifiers.

Definition 33. *Let E be a finite set, then for $q \in \mathbf{I}$ the function $\Delta_q : \widetilde{\mathcal{P}}(E)^2 \to \widetilde{\mathcal{P}}(E)$ is characterized element-wise by*

$$\mu_{\Delta_q(X_1,X_2)}(x) = (\mu_{X_1}(x) \vee (1-q)) \cdot \mu_{X_2}(x)^{\mu_{X_1}(x) \vee q}$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ and $x \in E$.

We now have all the definitions we need to state the OWA models as introduced by Yager in [18].

Definition 34 (OWA models)**.** *The OWA models are given by the following functionals*

- *Let $E \neq \emptyset$ be a finite set with $|E| = n$, then*

  $$\mathrm{OWA}^{(1)}_{\mathrm{rel}}(\mu_Q)(X_1) = O_{\mathbf{w}}(\mathbf{x})$$

  for all coherent $\mu_Q : \mathbf{I} \to \mathbf{I}$ and $X_1 \in \widetilde{\mathcal{P}}(E)$, where $O_{\mathbf{w}}$ is an n-dimensional OWA operator with $\mathbf{w}_i = \mu_{Q,E}(i) - \mu_{Q,E}(i-1)$ and $\mathbf{x}_i = \mu_{[i]}(X_1)$ for $i = \{1, \ldots, n\}$,

- *Let $E \neq \emptyset$ be a finite set with $|E| = n > 1$, then*

  $$\mathrm{OWA}^{(2)}_{\mathrm{rel}}(\mu_Q)(X_1, X_2) = (\mathrm{OWA}^{(1)}_{\mathrm{rel}}(\mu_Q) \circ \Delta_q)(X_1, X_2)$$

  for all coherent $\mu_Q : \mathbf{I} \to \mathbf{I}$ and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$.

The OWA method has several adequacy limitations that we will discuss now. First of all it is not applicable in the case of two-place quantification if the base set contains only one element, since the orness is not defined in that case. The second limitation is caused by the restriction of the OWA method to regular increasing quantifiers. Yager tried to solve this problem by introducing the so-called *quantifier synthesis method* [17], which consists in representing arbitrary fuzzy linguistic quantifiers by a boolean combination of regular increasing quantifiers. These regular increasing quantifiers are then translated by the OWA method to their fuzzy models, which in turn are combined by the same boolean combination in order to produce a fuzzy quantifier. However, in general there are several possible decompositions of a fuzzy linguistic quantifier, and some of them yield different results. Another inadequate behavior of the OWA method is its failure to preserve the property of having extension for some fuzzy quantifiers. This is demonstrated in the following example.

Example 3 (Non-extensionality of OWA [7]). Consider the universe $E = \{\mathrm{max}, \mathrm{susi}\}$, the sets **child** $= \emptyset$, **hungry** $= E$ and the linguistic quantifier $Q =$ **at least half**. Assume Q is modeled by the fuzzy linguistic quantifier μ_Q with

$$\mu_Q(x) = \begin{cases} 1 & \text{if } x \geq 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the evaluation of the sentence "at least half of the children are hungry" with the OWA approach yields

$$\mathrm{OWA}^{(2)}_{\mathrm{rel}}(\mu_Q)(\mathbf{child}, \mathbf{hungry}) = \mathrm{OWA}^{(1)}_{\mathrm{rel}}(\mu_Q)(\emptyset) = 0.$$

Now consider the extended universe $E' = E \cup \{\text{franz}\}$, then without any modification of **child** and **hungry** the evaluation of the same sentence results in the following

$$\begin{aligned}\mathrm{OWA}^{(2)}_{\mathrm{rel}}(\mu_Q)(\mathbf{child}, \mathbf{hungry}) &= \mathrm{OWA}^{(1)}_{\mathrm{rel}}(\mu_Q)(\{0.5/\text{max}, 0.5/\text{susi}, 0/\text{franz}\}) \\ &= 0.5.\end{aligned}$$

The fuzzy quantifier $\mathrm{OWA}^{(2)}_{\mathrm{rel}}(\mu_Q)$ does therefore not have extension. □

The results in the previous example are due to fact that the orness associated with μ_Q, varies depending on the cardinality of the universe. Indeed it holds that $\mathrm{orness}(\mu_{Q,E}) = 1$ and $\mathrm{orness}(\mu_{Q,E'}) = 0.5$ and thus $\Delta_1(\mathbf{child}, \mathbf{hungry}) = \emptyset$ and $\Delta_{0.5}(\mathbf{child}, \mathbf{hungry}) = \{0.5/\text{max}, 0.5/\text{susi}, 0/\text{franz}\}$.

Besides the previous inadequate behaviors it was observed by Glöckner in [7] that "all" and "some" are the only conservative semi-fuzzy quantifiers, which can be represented by the OWA-approach. This is formally stated in the following proposition.

Proposition 2. *Let E be a finite set with $|E| > 1$ and let $\mu_Q : \mathbf{I} \to \mathbf{I}$ such that* $\mathrm{orness}(\mu_{Q,E}) \in (0,1)$ *then* $\mathrm{OWA}^{(2)}_{\mathrm{rel}}(\mu_Q)|_{\mathcal{P}(E)^2}$ *is not conservative.*

Proof. $\mathrm{OWA}^{(2)}_{\mathrm{rel}}(\mu_Q)(\emptyset, \emptyset) = 0 \neq (1-q) = \mathrm{OWA}^{(2)}_{\mathrm{rel}}(\mu_Q)(\emptyset, E)$. □

2.3 FG-count approach

This approach is again based on a cardinality measure of fuzzy sets, namely the FG-count, which is a fuzzy measure that was initially presented by Zadeh in [20]. The FG-count method is meant to model fuzzy quantifiers based on increasing fuzzy linguistic quantifiers, even though it is well defined for any kind of fuzzy linguistic quantifier, as we will later see. The FG-count measure of a fuzzy set corresponds to a fuzzy subset of the natural numbers, which, intuitively gives the information to which degree the fuzzy set contains at least n elements. It is formally defined as follows.

Definition 35 (FG-count)**.** *Let E be a finite set and let $X \in \widetilde{\mathcal{P}}(E)$ then the FG-count of X, in symbols $\mathrm{FG}_{\mathrm{count}}(X) \in \widetilde{\mathcal{P}}(\mathbb{N})$, is characterized by*

$$\begin{aligned}\mu_{\mathrm{FG}_{\mathrm{count}}(X)}(i) &=_{\mathrm{def}} \sup\{\alpha \in \mathbf{I} : |X_{\geq\alpha}| \geq i\} \textit{ with } \sup\{\} = 0 \\ &= \mu_{[i]}(X).\end{aligned}$$

It is not very hard to see that the FG-count represents a more informative cardinality measure than the sigma-count, indeed the sigma-count constitutes a summary of the FG-count, since we can express it as follows $\Sigma_{\mathrm{count}}(X) = \sum_{i=1}^{n} \mu_{[j]}(X)$. We are now ready to state the definition of the FG-count models as they were initially introduced by Yager in [17].

Definition 36 (FG-count models)**.** *Let E be a finite base set then the FG-count models for absolute one- and two-place quantifiers are given by*

$$\mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_Q)(X_1) = \max\{\mu_Q(i) \wedge \mu_{\mathrm{FG}_{\mathrm{count}}(X_1)}(i) : i \in \{0, \dots, n\}\}$$
$$\mathrm{FG}^{(2)}_{\mathrm{abs}}(\mu_Q)(X_1, X_2) = \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_Q)(X_1 \cap X_2)$$

for all $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. The models for relative one-place quantifiers are defined as

$$\mathrm{FG}^{(1)}_{\mathrm{rel}}(\mu_Q)(X_1) = \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_{Q,E})(X_1)$$

for all $\mu_Q : \mathbf{I} \to \mathbf{I}$ and $X_1 \in \widetilde{\mathcal{P}}(E)$.

The case of two-place relative quantifiers is not covered by the FG-count approach. This limitation greatly reduces the practical relevance of this method since relative two-place quantification is fairly common in practical applications. There are some possible generalizations of the FG-count models to the two-place relative quantification, which are explained in [7]. These methods are however somewhat problematic. This is why we will not examine them here. As it was already mentioned earlier the FG-count approach can be applied to any fuzzy linguistic quantifier in order to produce a fuzzy quantifier, but the resulting quantifiers will always be increasing in all arguments.

Proposition 3. *Let $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ and $\mu'_Q : \mathbf{I} \to \mathbf{I}$ then $\mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_Q)$, $\mathrm{FG}^{(1)}_{\mathrm{rel}}(\mu'_Q)$ are increasing and $\mathrm{FG}^{(2)}_{\mathrm{abs}}(\mu_Q)$ is increasing in both arguments.*

This is why the FG-count approach is unfit to model any other quantifier than increasing ones. In order to overcome this limitation we could again try to apply the quantifier synthesis method, but this will as in the case of the OWA method produce different results depending on the chosen decomposition [7].

2.4 Sugeno integral and Choquet integral approaches

This section will briefly explain the Sugeno and Choquet integral approaches, which are of interest because of their relation to the FG-count and the OWA methods. Both methods were defined by Bosc and Liétard in [2], and are intended to operate only on increasing fuzzy linguistic quantifiers.

Definition 37 (Sugeno integral [8])**.** *Let (X, Ω, μ) be a fuzzy measure space with $X = \{x_1, ..., x_n\}$ then the Sugeno integral of a function $f : X \to \mathbf{I}$ is defined as*

$$\mathcal{S}_\mu(f) = \bigvee_{i=1}^{n} (f(x_{(i)}) \wedge \mu(A_{(i)}))$$

where the subscript (i) indicates a permutation of X such that $0 \le f(x_{(1)}) \le \dots \le f(x_{(n)}) \le 1$, and $A_{(i)} = \{x_i, \dots, x_n\}$.

Definition 38 (Sugeno integral models)**.** *Let $E \neq \emptyset$ be a finite set, then the Sugeno integral models for absolute one-place quantifiers are defined as*

$$\mathrm{SI}^{(1)}_{\mathrm{abs}}(\mu_Q)(X) = \mathcal{S}_{\mu_Q \circ |\cdot|}(\mu_X)$$

for all regular nondecreasing $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(E)$. *The models for relative one-place are given by*

$$\mathrm{SI}^{(1)}_{\mathrm{rel}}(\mu_Q)(X) = \mathrm{SI}^{(1)}_{\mathrm{abs}}(\mu_{Q,E})(X),$$

for all coherent $\mu_Q : \mathbf{I} \to \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(E)$.

The Sugeno integral method is well defined in the case of nondecreasing fuzzy linguistic quantifiers, however, in order to comply with the notion of regular fuzzy measure the fuzzy linguistic quantifiers were originally required to be regular. The next proposition is then an immediate consequence of the definitions of the Sugeno integral models and the FG-count models.

Proposition 4. *In the case of one-place quantification the FG-count models and the Sugeno integral models coincide.*

Proof. Let $E \neq \emptyset$ be a finite base set, $X \in \widetilde{\mathcal{P}}(E)$ and $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ be regular nondecreasing, then

$$\begin{aligned}
\mathrm{SI}^{(1)}_{\mathrm{abs}}(\mu_Q)(X) = \mathcal{S}_{\mu_Q \circ |\cdot|}(\mu_X) &= \bigvee_{i=1}^{n} (\mu_X(x_{(i)}) \wedge \mu_Q(n-i+1)) \\
&= \bigvee_{i=1}^{n} (\mu_{[n-i+1]}(X) \wedge \mu_Q(n-i+1)) \\
&= \bigvee_{j=1}^{n} (\mu_{[j]}(X) \wedge \mu_Q(j)) \\
&= \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_Q)(X)
\end{aligned}$$

□

This result is interesting since it allows us to explain the FG-count by the theory of fuzzy measures and fuzzy integrals. We will now examine the approach based on the Choquet integral. In order to do so we first need the general definition of the Choquet integral.

Definition 39 (Choquet integral [8])**.** *Let* (X, Ω, μ) *be a measurable space. Then the Choquet integral of a function* $f : X \to \mathbf{I}$ *is defined by*

$$\mathcal{C}_\mu(f) = \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)})) \cdot \mu(A_{(i)})$$

with the same notations as in definition 37.

We are now ready to define the Choquet integral approach for fuzzy quantification.

Definition 40 (Choquet integral models)**.** *Let* $E \neq \emptyset$ *be a finite set with* $|E| = n$*, then the models for absolute one-place quantifiers are given by*

$$\mathrm{CI}^{(1)}_{\mathrm{abs}}(\mu_Q)(X) = \mathcal{C}_{\mu_Q \circ |\cdot|}(\mu_X)$$

for any regular nondecreasing $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ *and all* $X \in \widetilde{\mathcal{P}}(E)$. *The models for relative one-place quantifiers are defined as*

$$\mathrm{CI}^{(1)}_{\mathrm{rel}}(\mu_Q)(X) = \mathrm{CI}^{(1)}_{\mathrm{abs}}(\mu_{Q,E})(X)$$

for all coherent $\mu_Q : \mathbf{I} \to \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(E)$.

The next proposition is then an immediate consequence of the previous definition and the definition of the OWA models.

Proposition 5. *In the case of relative one-place quantification the OWA models coincide with the Choquet integral models.*

Proof. Let $E \neq \emptyset$ be a finite base set with $|E| = n$, $X \in \widetilde{\mathcal{P}}(E)$ and $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$ be regular nondecreasing, then

$$\begin{aligned}
\mathrm{CI}^{(1)}_{\mathrm{abs}}(\mu_Q)(X) = \mathcal{C}_{\mu_Q \circ |\cdot|}(\mu_X) &= \sum_{i=1}^{n} (\mu_X(x_{(i)}) - \mu_X(x_{(i-1)})) \cdot \mu_Q(n-i+1) \\
&= \sum_{i=1}^{n} (\mu_{[n-i+1]}(X) - \mu_{[n-i+2]}(X)) \cdot \mu_Q(n-i+1) \\
&= \sum_{j=1}^{n} (\mu_{[j]}(X) - \mu_{[j+1]}(X)) \cdot \mu_Q(j) \\
&= \sum_{j=1}^{n} \mu_{[j]}(X)(\mu_Q(j) - \mu_Q(j-1))
\end{aligned}$$

Hence, for all coherent $\mu_{Q'} : \mathbf{I} \to \mathbf{I}$ and all $X' \in \widetilde{\mathcal{P}}(E)$ we have,

$$\begin{aligned}
\mathrm{CI}^{(1)}_{\mathrm{rel}}(\mu_{Q'})(X') &= \mathrm{CI}^{(1)}_{\mathrm{abs}}(\mu_{Q',E})(X') \\
&= \sum_{j=1}^{n} \mu_{[j]}(X')(\mu_{Q',E}(j) - \mu_{Q',E}(j-1)) \\
&= \mathrm{OWA}^{(1)}_{\mathrm{rel}}(\mu_{Q'})(X').
\end{aligned}$$

□

Both, the Sugeno integral method and the Choquet integral method can be generalized to the case of infinite base sets. Such generalizations have been proposed by Ying in [19] and by Cui and Li in [3].

2.5 The $\mathcal{G}$-family approach

In [4] Delgado et al. present a whole family of evaluation methods based on the family $\mathcal{E}$ of cardinalities of fuzzy sets defined below. The so-called $\mathcal{G}$-family of evaluation methods is mainly intended to model the case of absolute and relative one-place quantification, however, the authors also give generalizations of chosen methods to the case of two-place quantification. The methods of the $\mathcal{G}$-family extend most of the previously explained methods and are, therefore, very interesting from a theoretical point of view.

In order to understand the $\mathcal{G}$-family approach, we must define the $\mathcal{L}$-family, that is a family of cardinality measures of fuzzy sets based on t-norms and t-conorms.

Definition 41 ($\mathcal{L}$-family)**.** *Let E be a finite base set with $|E| = n$, then the*

$\mathcal{L}$-family is the set of functions $\lambda_X^{\top,\bot} : \mathbb{N} \to \mathbf{I}$ *with*

$$\lambda_X^{\top,\bot}(j) = \begin{cases} 0 & \text{if } j = 0 \\ 1 & \text{if } j > |E| \\ \underset{(i_1,\ldots,i_j)\in I_j}{\bot}(\overset{j}{\underset{k=1}{\top}} \mu_X(x_{i_k})) & \text{if } 1 \le j \le |E| \end{cases}$$

where $X \in \widetilde{\mathcal{P}}(E)$, $\top$ *is any t-norm,* $\bot$ *is any t-conorm and* $I_j = \{(i_1, \ldots, i_j) \in \{1, \ldots, n\}^j : i_m < i_{m+1}, 1 \le m < j\}$.

The functions of the $\mathcal{L}$-family may also be understood as fuzzy subsets of the natural numbers that express the degree to which a given set X contains at least j elements. This family of fuzzy cardinality counts is then used to define another family of cardinality counts, namely the $\mathcal{E}$-family, whose functions characterize fuzzy sets of the natural numbers that represent the possibility that a given fuzzy set X contains exactly j elements.

Definition 42 ($\mathcal{E}$-family)**.** *Let E be a finite base set, then the $\mathcal{E}$-family is the set of functions* $\varepsilon_X^{\top,\widetilde{\neg}} : \mathbb{N} \to \mathbf{I}$ *with*

$$\varepsilon_X^{\top,\widetilde{\neg}}(j) = \top(\lambda_X(j), \widetilde{\neg}\lambda_X(j+1)),$$

where $X \in \widetilde{\mathcal{P}}(E)$, $\top$ *is any t-norm,* $\widetilde{\neg}$ *is a fuzzy negation and* $\lambda \in \mathcal{L}$.

We are are now ready to formalize the class $\mathcal{G}$ of evaluation methods.

Definition 43 ($\mathcal{G}$ models)**.** *Let E be a finite base set with $|E| = n$, then the $\mathcal{G}$-family consists of the functions given below for the absolute and the relative case, respectively:*

$$\mathrm{G}_{\mathrm{abs}}^{(1),\top,\bot}(\mu_Q)(X) = \overset{n}{\underset{i=0}{\bot}}(\top(\varepsilon_X(i), \mu_Q(i)),$$

for all $\mu_Q : \mathbb{R}^+ \to \mathbf{I}$, $X \in \widetilde{\mathcal{P}}(E)$, *any t-norm* $\top$, *any t-conorm* $\bot$ *and any* $\varepsilon \in \mathcal{E}$.

$$\mathrm{G}_{\mathrm{rel}}^{(1),\top,\bot}(\mu_Q)(X) = \mathrm{G}_{\mathrm{abs}}^{(1),\top,\bot}(\mu_{Q,E})(X),$$

for all $\mu_Q : \mathbf{I} \to \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(E)$, *any t-norm* $\top$ *and any t-conorm* $\bot$.

The authors of this family identify furthermore two particular methods called the GD and GZ methods. These methods are interesting in that they generalize the Choquet integral method and the Sugeno integral method, respectively [4].

2.6 Representational level approach

Another interesting approach is presented in [14]. This approach differs from the previous ones in that it uses representational levels to model vagueness. A representational level consists of a set of levels that are associated each with a crisp subset of the base set. Formally, a representational level (RL) can be defined as follows.

Definition 44 (Representational level)**.** *Let E be a base set then an RL is a pair $X = (\Lambda, \rho)$ with $\Lambda \in \mathcal{P}(\mathbf{I})$ and $\rho : \Lambda \to \mathcal{P}(E)$. We call $\Omega_X = \{\rho(\alpha) : \alpha \in \Lambda\}$ the set of crisp representatives of X.*

Obviously RLs are defined in the case of the base set being infinite and possibly continuous. In the following we will, however, restrict our attention to the finite case since the method in [14] is also restricted to the finite case. We can now define the RL representation of a fuzzy set.

Definition 45. *Let E be a finite base set. Then the RL representation of $X \in \widetilde{\mathcal{P}}(E)$ is the pair (Λ_X, ρ_X) with*

$$\Lambda_X = \{\mu_X(e) : e \in \text{support}(X)\} \cup \{1\} \text{ and } \rho_X(\alpha) = X_{\geq \alpha}.$$

The (fuzzy) set operations union, intersection and complementation can be extended to RLs, but these are usually defined level-wise. This can be problematic since the level-wise complementation of an RL representing a fuzzy set will in general not result in an RL representing a fuzzy set.

Before we can proceed to the definition of the RL approach, we need to define the compatibility of two fuzzy sets in terms of their RL representations.

Definition 46. *Let A, B be fuzzy sets and let $(\Lambda_A, \rho_A), (\Lambda_B, \rho_B)$ be their respective RLs then the compatibility of A to B (B/A) is the RL $(\Lambda_{B/A}, \rho_{B/A})$ with $\Lambda_{B/A} = \Lambda_B \cup \Lambda_A$ and*

$$\rho_{B/A}(\alpha) = \frac{|\rho_A(\alpha) \cap \rho_B(\alpha)|}{|\rho_A(\alpha)|}, \text{ for all } \alpha \in \Lambda_{B/A}.$$

It should be noticed that the value of $\rho_{B/A}(\alpha)$ is not defined in the case of $\rho_A(\alpha) = \emptyset$, this case has been silently ignored in the literature. Any models based on this compatibility measure are therefore not applicable in the case of the quantifier restriction A being a non-normalized fuzzy set, i.e. if $1 \notin \hat{\mu}_A(E)$. We are now ready to define the first part of the RL evaluation method for the case of relative two-place quantification.

Definition 47 (RL models)**.** *Let E be a finite base set, then the models for relative two-place quantification are defined as*

$$\text{rl}^{(2)}_{\text{rel}}(\mu_Q)(X_1, X_2) = (\Lambda_{X_2/X_1}, \mu_Q \circ \rho_{X_2/X_1}),$$

for all $\mu_Q : \mathbf{I} \to \mathbf{I}$, all normalized $X_1 \in \widetilde{\mathcal{P}}(E)$ and all $X_2 \in \widetilde{\mathcal{P}}(E)$.

This approach yields a RL instead of a truth value and is therefore not entirely compatible with our idea of fuzzy quantifiers. We must thus use a summarization function to compute a truth value corresponding to the resulting RL. A concrete instantiation of this approach based on a probabilistic summarization method is given in [14]. The method uses the following probability density p_X over the crisp representatives of an RL X to realize the aggregation.

Definition 48. *Let $X = (\Lambda_X, \rho_X)$ be an RL with $\Lambda_X = \{\alpha_1, \ldots, \alpha_m\}$ such that $1 = \alpha_1 > \cdots > \alpha_m > \alpha_{m+1} = 0$. Then the probability density $p_X : \Omega_X \to \mathbf{I}$ is given by*

$$p_X(Y) = \sum_{\alpha_i \in \Lambda_X : Y = \rho_X(\alpha_i)} \alpha_i - \alpha_{i+1}, \text{ for all } Y \in \Omega_X$$

We are now ready to formalize the evaluation method given by Sánchez et al. in [14], which corresponds to the expected value of the quantification results produced by $\text{rl}^{(2)}_{\text{rel}}$.

Definition 49 (RL models)**.** *Let E be a finite base set, then the RL models for the case of relative two-place quantification is given by*

$$\mathrm{RL}^{(2)}_{\mathrm{rel}}(\mu_Q)(X_1, X_2) = \sum_{\beta \in \Omega_\Xi} \rho_\Xi(\beta) \cdot \beta, \;\; \textit{with } \Xi = \mathrm{rl}^{(2)}_{\mathrm{rel}}(\mu_Q)(X_1, X_2),$$

for all $\mu_Q : \mathbf{I} \to \mathbf{I}$, all normalized $X_1 \in \widetilde{\mathcal{P}}(E)$ and all $X_2 \in \widetilde{\mathcal{P}}(E)$.

This concrete RL method coincides with a particular case of the somewhat similar method introduced by Liétard and Rocacher in [11], in which the quantification is represented as a gradual truth value i.e. a fuzzy subset of $\mathbf{I}$ that is subsequently reduced to a truth value.

Because of their different representation the methods based on representational levels are mostly incompatible with the definitions of formal adequacy criteria used for the classical methods. Therefore, this approach has not yet been systematically analyzed for adequacy.

3 Methods based on semi-fuzzy quantifiers

Besides the approaches based on fuzzy linguistic quantifiers that we have considered in the previous section, there is another important family of more recent approaches that use semi-fuzzy quantifiers as the specification of fuzzy quantifiers. From the perspective of generality the use of semi-fuzzy quantifiers has several advantages over fuzzy linguistic quantifiers. Semi-fuzzy quantifiers can take any number of arguments that range over subsets of the base set and are, therefore, structurally much more similar to fuzzy quantifiers. This structural similarity allows for a more systematic projection of semi-fuzzy quantifiers into the space of fuzzy quantifiers. Furthermore, the specification of adequacy criteria for methods using semi-fuzzy quantifiers is more straightforward (see 3.2.1).

3.1 Representational levels

The representational level approach described above can be extended by means of semi-fuzzy quantifiers in a straightforward manner to the case of multi-place quantification [13]. We will only have a brief overview since the method is very similar to that which uses fuzzy linguistic quantifiers.

Definition 50 (RL models)**.** *Let $E \neq \emptyset$ and $Q : \mathcal{P}(E)^n \to \mathbf{I}$ be an n-ary semi-fuzzy quantifier. Then the RL model for n-place quantification is given by*

$$\mathrm{RL}_2(Q)(X_1, \dots, X_n) = (\bigcup_{i=1}^{n} \Lambda_{X_i}, Q \circ \mathop{\times}_{i=1}^{n} \rho_{X_i})$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, where $(\Lambda_{X_1}, \rho_{X_1}), \dots, (\Lambda_{X_n}, \rho_{X_n})$ are their respective RL representations.

Again the result is an RL which needs to be summarized to a fuzzy truth value in order to obtain a fuzzy quantifier according to our definition. There are various methods to accomplish this, one of them can be found in [13].

3.2 Quantifier fuzzification mechanisms

In [7] Glöckner introduces a very general approach to fuzzy quantification, which is based on the so-called quantifier fuzzification mechanisms, or QFMs for short. The concept of a QFM corresponds to a mapping that maps semi-fuzzy quantifiers to fuzzy quantifiers. Formally a QFM is as defined below.

Definition 51 (Quantifier fuzzification mechanism)**.** *A quantifier fuzzification mechanism is a mapping* $\mathcal{F} : (\mathcal{P}(E)^n \to \mathbf{I}) \to (\widetilde{\mathcal{P}}(E)^n \to \mathbf{I})$ *that assigns to each semi-fuzzy quantifier* Q *a fuzzy quantifier* $\mathcal{F}(Q)$*.*

So far QFMs are completely unrestricted and any mapping that maps any semi-fuzzy quantifier to a fuzzy quantifier is a QFM, but obviously there are many QFMs that do not represent adequate translations schemes. Therefore, Glöckner introduced the concept of *determiner fuzzification scheme* (DFS), which are QFMs that respect certain adequacy constraints that are considered to be necessary to produce linguistic adequate fuzzy quantifiers.

3.2.1 Introduction to DFS-Theory

We will now concern ourselves with the concept of determiner fuzzification schemes which restricts the set of plausible QFMs. Before we can see the axiomatic definition of a DFS we must consider several concepts related to QFMs.

First of all we need to consider the concept of projection quantifiers, which in the crisp case corresponds to a predicate that given an element and a set returns true if the element belongs to that set. We can extend this behavior naturally to fuzzy sets by using the membership degree of the element and the given set.

Definition 52 (Projection quantifier)**.** *Let* E *be the base set and* $e \in E$*. We define the projection quantifier* $\pi_e : \mathcal{P}(E) \to \mathbf{I}$ *by*

$$\pi_e(Y) = \chi_Y(e),$$

for all $Y \in \mathcal{P}(E)$*. Analogously we define the fuzzy projection quantifier* $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \to \mathbf{I}$ *by*

$$\widetilde{\pi}_e(X) = \mu_X(e),$$

for all $X \in \widetilde{\mathcal{P}}(E)$*.*

One problem with fuzzy logic is the possibility for truth functions to have several possible plausible extensions, however not every extension is perfectly compatible with the way we interpret the fuzzy quantifiers. QFMs have the nice property of inducing truth functions, which solves the problem of choosing an adequate set of connectives. In order to define the induced truth functions it is useful to interpret n-tuples of truth values as (fuzzy) subsets of the numbers $\{1, \ldots, n\}$.

Definition 53. *We define the mapping* $\eta : \mathbf{2}^n \to \mathcal{P}(\{1, \ldots, n\})$ *by*

$$\eta(b_1, \ldots, b_n) = \{k \in \{1, \ldots, n\} : b_k = 1\},$$

for all $b_1, \ldots, b_n \in \mathbf{2}$*. Furthermore, the mapping* $\widetilde{\eta} : \mathbf{I}^n \to \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ *is characterized by*

$$\mu_{\widetilde{\eta}(x_1, \ldots, x_n)}(k) = x_k,$$

for all $x_1, \dots, x_n \in \mathbf{I}$ *and all* $k \in \{1, \dots, n\}$.

We are now ready to formulate the concept of induced truth function. The definition uses the previously defined bijective mapping η to represent semi-fuzzy truth functions as a semi-fuzzy quantifier which is then translated by means of the QFM to a fuzzy quantifier. The induced truth function is then obtained by composing the previously obtained fuzzy quantifier with the function $\widetilde{\eta}$.

Definition 54 (Induced truth functions)**.** *Let* $\mathcal{F}$ *be a QFM,* $n \in \mathbb{N}$ *and* $f : \mathbf{2}^n \to \mathbf{I}$*. Then, the semi-fuzzy quantifier* $Q_f : \mathcal{P}(\{1, \dots, n\}) \to \mathbf{I}$ *is given by*

$$Q_f(X) = f(\eta^{-1}(X)),$$

for all $X \in \mathcal{P}(\{1, \dots, n\})$*. The induced truth function* $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \to \mathbf{I}$ *is defined in terms of* Q_f

$$\widetilde{\mathcal{F}}(f)(x_1, \dots, x_n) = \mathcal{F}(Q_f)(\widetilde{\eta}(x_1, \dots, x_n)),$$

for all $x_1, \dots, x_n \in \mathbf{I}$.

In the context of a given QFM $\mathcal{F}$ we denote by $\widetilde{\neg}, \widetilde{\wedge}, \widetilde{\vee}$ the induced fuzzy connectives i.e. $\widetilde{\neg} = \mathcal{F}(\neg)$, $\widetilde{\wedge} = \mathcal{F}(\wedge)$ and $\widetilde{\vee} = \mathcal{F}(\vee)$. Furthermore, we denote by $\widetilde{\neg}$, $\widetilde{\cap}$, $\widetilde{\cup}$ the "induced" fuzzy set operations defined in terms of the induced fuzzy connectives. For instance, the induced fuzzy complement is characterized by

$$\widetilde{\neg} X = \widetilde{\neg} \mu_X(e),$$

for all $e \in E$ and all $X \in \widetilde{\mathcal{P}}(E)$.

The definition of a DFS will also be concerned about how a QFM behaves with functional application on the arguments, this means we will need to talk about how a QFM transforms functions that are applied to the arguments of a semi-fuzzy quantifier. We, therefore, introduce the notion of induced extension principle. As in the case of induced truth functions the notion of an induced extension principle avoids us to manually choose an extension principle which is compatible with the given QFM.

Definition 55 (Induced extension principle)**.** *Assume* $\mathcal{F}$ *is a QFM, then its induced extension principle* $\widehat{\mathcal{F}}$ *assigns to each* $f : E \to E'$ *with* $E, E' \neq \emptyset$ *a mapping* $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \to \widetilde{\mathcal{P}}(E')$ *characterized by*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\pi_{e'} \circ \hat{f})(X),$$

for all $X \in \widetilde{\mathcal{P}}(E)$ *and* $e' \in E'$.

We are now ready to formulate the compatibility with functional application, which requires a QFM to generate the same fuzzy quantifiers regardless of whether the function is applied before fuzzification or if its fuzzy counterpart is applied to the fuzzified quantifier.

Definition 56. *Let* $g_1, \dots, g_n : A \to B$ *then we denote by* $\bigtimes_{i=1}^{n} g_i : A^n \to B^n$ *the product mapping which is defined by*

$$\bigtimes_{i=1}^{n} g_i(x_1, \dots, x_n) = (g_1(x_1), \dots, g_n(x_n)),$$

whenever $g_1(x_1), \dots, g_n(x_n)$ *are defined.*

Definition 57 (Compatibility to functional application)**.** *We say that a QFM $\mathcal{F}$ is compatible with functional application, if it satisfies*

$$\mathcal{F}(Q \circ \bigtimes_{i=1}^{n} \hat{f}_i) = \mathcal{F}(Q) \circ \bigtimes_{i=1}^{n} \widehat{\mathcal{F}}(f_i),$$

for all $Q : \mathcal{P}(E) \to \mathbf{I}$ and all $f_1, \ldots, f_n : E' \to E$ with $E' \neq \emptyset$.

We have now seen all definitions that are necessary to provide a concise axiomatization for the concept of quantifier fuzzification schemes.

Definition 58 (Determiner fuzzification scheme)**.** *A QFM $\mathcal{F}$ is called a determiner fuzzification scheme – DFS for short – if the following conditions are satisfied for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \to \mathbf{I}$*

$\mathcal{F}(Q)\vert_{\mathcal{P}(E)^n} = Q,$ *if $n \leq 1$,*	(Correct generalization)
$\mathcal{F}(Q) = \widetilde{\pi}_e,$ *if $Q = \pi_e$ for some $e \in E$,*	(Projection quantifiers)
$\mathcal{F}(Q\widetilde{\Box}) = \mathcal{F}(Q)\widetilde{\Box}, n > 0$	(Dualization)
$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup}, n > 0$	(Internal joins)
if Q is decreasing in the n-th argument, then $\mathcal{F}(Q)$ is decreasing in the n-th argument, $n > 0$	(Preservation of monotonicity)
$\mathcal{F}$ is compatible with functional application.	(Functional application)

Note that the axioms above are independent of each other, i.e. no axiom is redundant. It can be shown that they properly express the corresponding adequacy constraints. Moreover the DFS axioms entail many other useful properties like e.g. the compatibility with argument transposition, cylindrical extensions, the formation of antonyms, quantifier negation, argument intersections, etc. for more details see [7].

Even though the DFS axioms considerably restrict the space of plausible QFMs, we need further restrictions to find the practically relevant DFSes. Therefore we classify the DFSes according to their induced fuzzy disjunction and negation.

Definition 59 (($\widetilde{\neg}, \widetilde{\vee}$)-DFSes)**.** *A DFS $\mathcal{F}$ such that $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ is called a ($\widetilde{\neg}, \widetilde{\vee}$)-DFS.*

The practically most interesting DFSes are those that induce the standard fuzzy connectives.

Definition 60 (Standard DFSes)**.** *Any ($\neg, \vee$)-DFS is also called a standard DFS.*

3.2.2 Glöckner's QFMs

In the following we will consider an important class of DFSes investigated by Glöckner in [7] and understand its structure. In particular we will consider the construction of the $\mathcal{M}_{CX}$-DFS which has unique adequacy properties.

The first class of QFMs analyzed by Glöckner is based on a three valued cut depending on a parameter γ, also called the uncertainty, which defines the range of the cut.

Definition 61 (Three-valued ambiguity range)**.** *Let E be a set, $X \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$ then we define $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ and $X_\gamma^{\min}, X_\gamma^{\max} \in \mathcal{P}(E)$ by*

$$\mathcal{T}_\gamma(X) = \{Y : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\},$$

$$X_\gamma^{\min} = \begin{cases} X_{\geq \frac{1}{2}+\frac{1}{2}\gamma}, & \text{if } \gamma > 0 \\ X_{>\frac{1}{2}}, & \text{otherwise,} \end{cases}$$

$$X_\gamma^{\max} = \begin{cases} X_{>\frac{1}{2}-\frac{1}{2}\gamma}, & \text{if } \gamma > 0 \\ X_{\geq\frac{1}{2}}, & \text{otherwise.} \end{cases}$$

Intuitively, the ambiguity range $\mathcal{T}_\gamma(X)$ can be thought of as the set of crisp representatives of the fuzzy set X, where γ is the tolerance up to which an element with membership degree below $\frac{1}{2}$ can be part of a crisp representative.

The sets in the ambiguity range are then evaluated by the semi-fuzzy quantifier and finally the quantification results are aggregated using the following fuzzy median in order to obtain a fuzzy quantifier for a given uncertainty level γ.

Definition 62 (Fuzzy medians)**.** *We define the fuzzy median $\operatorname{med}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \to \mathbf{I}$ by*

$$\operatorname{med}_{\frac{1}{2}}(u_1, u_2) = \begin{cases} \min(u_1, u_2), & \text{if } \min(u_1, u_2) > \frac{1}{2} \\ \max(u_1, u_2), & \text{if } \max(u_1, u_2) < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

for all $u_1, u_2 \in \mathbf{I}$. We define furthermore the generalized fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \to \mathbf{I}$ by

$$\mathrm{m}_{\frac{1}{2}}(X) = \operatorname{med}_{\frac{1}{2}}(\inf X, \sup X)$$

for all $X \in \mathcal{P}(\mathbf{I})$.

Definition 63. *Let $\gamma \in \mathbf{I}$, then we define the QFM $(\cdot)_\gamma$ by*

$$Q_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}(\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i), i \in \{1, \ldots, n\}\})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \to \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Finally, by aggregating the results for every uncertainty level using an integral we obtain the QFM defined below.

Definition 64. *We define the quantifier fuzzification mechanism $\mathcal{M}$ as follows*

$$\mathcal{M}(Q)(X_1, \ldots, X_n) = \int_0^1 Q_\gamma(X_1, \ldots, X_n) d\gamma$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \to \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Glöckner then found the following interesting result about the QFM $\mathcal{M}$.

Theorem 1. *$\mathcal{M}$ is a standard DFS.*

From there on Glöckner observed that it is possible to define a whole family of QFMs by simply replacing the integral in the definition of $\mathcal{M}$ by suitable aggregation operators. In order to be able to formulate this family of QFMs it is necessary to determine on which domain the aggregation operators can work on. It turned out that the set $\mathbb{B}$ defined below contains all possible functions that may arise by applying the QFM $(\cdot)_\gamma$ to any semi-fuzzy quantifier.

Definition 65. *The sets* $\mathbb{B}, \mathbb{B}^+, \mathbb{B}^{\frac{1}{2}}, \mathbb{B}^- \subseteq \mathbf{I}^{\mathbf{I}}$ *are defined by*

$$\begin{aligned}
\mathbb{B}^+ &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) > \frac{1}{2}, \hat{f}(\mathbf{I}) \subseteq \left[\frac{1}{2}, 1\right], f \text{ is decreasing}\}, \\
\mathbb{B}^{\frac{1}{2}} &= \{f \in \mathbf{I}^{\mathbf{I}} : f(x) = \frac{1}{2} \text{ for all } x \in \mathbf{I}\} \\
\mathbb{B}^- &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) < \frac{1}{2}, \hat{f}(\mathbf{I}) \subseteq \left[0, \frac{1}{2}\right], f \text{ is increasing}\}, \\
\mathbb{B} &= \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^-
\end{aligned}$$

The family of QFMs referred to above is called the $\mathcal{M}_{\mathcal{B}}$-family, it is formally defined as follows.

Definition 66 ($\mathcal{M}_{\mathcal{B}}$-QFMs)**.** *Let* $\mathcal{B} : \mathbb{B} \to \mathbf{I}$ *then we denote by* $\mathcal{M}_{\mathcal{B}}$ *the QFM defined by*

$$\mathcal{M}_{\mathcal{B}}(Q)(X_1, \ldots, X_n) = \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}})$$

for all $Q : \mathcal{P}(E)^n \to \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*.*

In [7] Glöckner analyses the properties of the whole class of $\mathcal{M}_{\mathcal{B}}$-QFMs and comes to the conclusion that every $\mathcal{M}_{\mathcal{B}}$-QFM is a standard DFS.

Theorem 2. *Every* $\mathcal{M}_{\mathcal{B}}$*-QFM is a standard DFS.*

Glöckner furthermore analyzed several individual DFSes of this class and found one QFM, namely $\mathcal{M}_{CX}$, which exhibits a unique behavior amongst all standard DFSes. Indeed it is the only standard model that is compatible with fuzzy argument insertion and moreover it is suspected to be the only standard model which weakly preserves convexity, for more details about these properties see [7].

Definition 67. *By* $\mathcal{M}_{CX}$ *we denote the* $\mathcal{M}_{\mathcal{B}}$*-QFM defined by*

$$\mathcal{B}_{CX}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sup\{\min(x, 2 \cdot f(x) - 1) : x \in \mathbf{I}\}, & \text{if } f \in \mathbb{B}^+ \\ \frac{1}{2}, & \text{if } f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\sup\{\min(x, 1 - 2 \cdot f(x)) : x \in \mathbf{I}\}, & \text{if } f \in \mathbb{B}^- \end{cases}$$

for all $f \in \mathbb{B}$*.*

It was found by Glöckner that the $\mathcal{M}_{CX}$ model properly generalizes the previously stated method based on the Sugeno integral.

According to Glöckner the model $\mathcal{M}_{CX}$ constitutes the best model of fuzzy quantification with regard to linguistic adequacy. Nevertheless, he continued the exploration of other classes of DFSes in order fo find more theoretically relevant models with greater discriminatory force. One of the classes investigated by Glöckner is the class $\mathcal{F}_\xi$, which arises from a generalization of the family $\mathcal{M}_{\mathcal{B}}$. This class $\mathcal{F}_\xi$ was found to contain a particular method, namely $\mathcal{F}_{Ch}$, which generalizes the Choquet integral method mentioned in section 2.4 [7].

3.2.3 Díaz-Hermida's QFMs

Díaz-Hermida et al. introduce in [6] a new family of QFMs based on a probabilistic voting-model. As observed by the authors the QFMs defined in the following are only well defined in the case of a finite base set, thus, strictly speaking these models are not QFMs but because of their similarity to Glöckners idea of QFMs we will call them finite QFMs.

A $\mathcal{F}^P$-QFM can be seen as an expected value of the semi-fuzzy quantifier over a given joint probability distribution of the α-cut levels of the arguments.

Definition 68. *Let E be a finite base set $P : \mathbf{I}^n \to \mathbf{I}$ be a probability density function then we denote by $\mathcal{F}^P$ the finite QFM defined by*

$$\begin{aligned}&\mathcal{F}^P(Q)(X_1, \dots, X_n)\\ &\quad = \int_0^1 \cdots \int_0^1 Q((X_1)_{\geq \alpha_1}, \dots, (X_n)_{\geq \alpha_n}) P(\alpha_1, \dots, \alpha_n) d\alpha_n \dots d\alpha_1\end{aligned}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \to \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

The first finite QFM defined by the authors is based on the maximum dependence model, which assumes that all voters choose the same focal element.

Definition 69. *By $\mathcal{F}^{MD}$ we denote the finite $\mathcal{F}^P$-QFM defined in terms of P^{MD} with*

$$P^{MD}(\alpha_1, \dots, \alpha_n) = \begin{cases} 1 & \text{if } \alpha_1 = \dots = \alpha_n \\ 0 & \text{otherwise} \end{cases}$$

for all $\alpha_1, \dots, \alpha_n \in \mathbf{I}$. We therefore have

$$\mathcal{F}^{MD}(Q)(X_1, \dots, X_n) = \int_0^1 Q((X_1)_{\geq \alpha}, \dots, (X_n)_{\geq \alpha}) d\alpha$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \to \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

The second particular $\mathcal{F}^P$-QFM is the so-called independence model which assumes that voters are perfectly independent in their choice of the focal element.

Definition 70. *By $\mathcal{F}^I$ we denote the finite $\mathcal{F}^P$-QFM defined by $P^I(\alpha_1, \dots, \alpha_n) = 1$ for all $\alpha_1, \dots, \alpha_n$, i.e.*

$$\mathcal{F}^I(Q)(X_1, \dots, X_n) = \int_0^1 \cdots \int_0^1 Q((X_1)_{\geq \alpha_1}, \dots, (X_n)_{\geq \alpha_n}) d\alpha_n \dots d\alpha_1$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \to \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

The authors furthermore defined another model $\mathcal{F}^{AD}$ called the approximate dependence model, which is based on the assumption that voters select approximately the same focal elements for different properties. Since this models needs to relate the arguments of the semi-fuzzy quantifier and depends on several parameters it is much more complicated to define, this is why we will not consider its definition. The interested reader can find a definition for the case of two-place quantification in [6].

The analysis of these models by Díaz-Hermida et al. in [6] revealed that the maximum dependency and independence models exhibit an adequate behavior. However, even in the finite case these QFMs do not represent DFSes, since the maximum dependency model fails to satisfy the dualization property, whereas the independence model fails to satisfy the internal joins.

4 Polyadic fuzzy quantification

So far we have only considered methods that are able to deal with the case of monadic quantification i.e. quantification over (fuzzy) subsets of the base set. But there are cases where a quantifier may not only bind subsets but also arbitrary relations over the base set. This is for example the case, if we try to model the meaning of sentences like (a) "Most men and most women like each other" and (b) "At least three girls gave more roses than lilies to John". These sentences might be represented as the polyadic Lindström quantifiers

$$Q'_{x,y,xy}(man(x), woman(y), like(x,y))$$
$$Q''_{x,y,z,xyz}(girl(x), rose(y), lily(z), give(x,y,z))$$

of type $\langle 1,1,2\rangle$ and $\langle 1,1,1,3\rangle$, respectively [7, 10].

In the literature there is yet only one notable approach to fuzzy polyadic quantification. This approach was introduced by Glöckner, who was particularly interested in the case of branching quantification [7]. Branching quantification occurs, if several quantifiers operate in parallel and independently of each other. To illustrate this, consider again the sentence (a) given above. It seems that two independent quantifiers are required to capture the mutuality of the concept expressed in this sentence. Indeed it is commonly believed that branching quantification is required to capture the meaning of propositions involving predicates that express reciprocity [7]. Glöckner argues that without branching quantification the sentence must be expressed as a linear succession of quantifiers, which essentially allows for the following two possibilities:

$$most(man)\ \hat{x}[most(woman)\ \hat{y}[like(x,y)]],$$
$$most(woman)\ \hat{y}[most(man)\ \hat{x}[like(x,y)]].$$

These representations of (a), however, do not capture the intended symmetry for they rather represent the meaning "Most men like most women" and vice-versa. Not even the conjunction of both formulas would capture the intended mutuality. Glöckner therefore suggests the use of Lindström quantifiers to handle the case of branching quantification and proposes the following type $\langle 1,1,2\rangle$ quantifier to represent the sentence (a) (assuming the crisp case for simplicity)

$$Q(A,B,R) = \begin{cases} 1 & \text{if } \exists U \times V \subseteq R : Q_{\mathbf{most}}(A,U) \wedge Q_{\mathbf{most}}(B,V) \\ 0 & \text{otherwise.} \end{cases}$$

In the following we will consider Glöckner's approach to fuzzy polyadic quantification which principally consists of a straightforward extension of his QFM framework. For brevity we will omit most of the details which can be found in [7], the essence of the approach should nevertheless be easily understood. Glöckner starts by providing a generalization of (semi-)fuzzy quantifiers, the so-called (semi-)fuzzy L-quantifiers. The prefix "L" indicates that these quantifiers are a generalized form of Lindström quantifiers which in their original form were not suited to linguistic applications.

Definition 71 ((Semi-)fuzzy L-quantifiers)**.** *A semi-fuzzy L-quantifier Q of type $\langle t_1, \ldots, t_n \rangle$ on a base set $E \neq \emptyset$ is a mapping $Q : \times_{i=1}^{n} \mathcal{P}(E^{t_i}) \to \mathbf{I}$, that assigns to each choice of arguments $Y_1 \in \mathcal{P}(E^{t_1}), \ldots, Y_n \in \mathcal{P}(E^{t_n})$ a result $Q(Y_1, \ldots, Y_n) \in \mathbf{I}$. A fuzzy L-quantifier $\widetilde{Q}$ of type $\langle t_1, \ldots, t_n \rangle$ on a base set $E \neq \emptyset$ is a mapping $\widetilde{Q} : \times_{i=1}^{n} \widetilde{\mathcal{P}}(E^{t_i}) \to \mathbf{I}$.*

The concept of QFM is now easily extended to the case of polyadic quantification.

Definition 72 (L-QFMs)**.** *An L-quantifier fuzzification mechanism $\mathcal{F}$ assigns to each semi-fuzzy L-quantifier of type $\langle t_1, \ldots, t_n \rangle$ a fuzzy L-quantifier $\mathcal{F}(Q)$ of the same type.*

Then, analogous to the monadic case it is possible to define the concept of L-DFS which restricts the space of L-QFMs to the plausible ones. To do so Glöckner found that every L-QFM $\mathcal{F}$ has an ordinary QFM denoted by $\mathcal{F}_R$ which he used to define the induced truth functions and the induced extension principle of L-QFMs.

Definition 73 (L-DFS)**.** *A L-QFM $\mathcal{F}$ is called an L-DFS if the following conditions are satisfied for every semi-fuzzy quantifier $Q : \mathop{\times}\limits_{i=1}^{n} \widetilde{\mathcal{P}}(E^{t_i}) \to \mathbf{I}$ of arbitrary types $t = \langle t_1, \ldots, t_n \rangle$ and for all base sets $E \neq \emptyset$:*

$$\mathcal{U}(F(Q)) = Q, \text{ if } t \in \{\langle\rangle, \langle 1 \rangle\}, \quad \text{(Correct generalization)}$$

$$\mathcal{F}(Q) = \widetilde{\pi}_{(e)}, \text{ if } Q = \pi_{(e)}, \quad \text{(Projection quantifiers)}$$

$$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square}, n > 0 \quad \text{(Dualization)}$$

$$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup}, n > 0 \quad \text{(Internal joins)}$$

if Q is decreasing in the n-th argument, (Preservation of monotonicity)
then $\mathcal{F}(Q)$ is decreasing in the n-th argument, $n > 0$

$\mathcal{F}$ is compatible with functional application. (Functional application)

Because the development of useful models required considerable effort, Glöckner searched for a method that allows to construct L-DFSes from ordinary DFSes. Indeed it turned out that such a construction exists. The L-DFS constructed from a DFS $\mathcal{F}$ is then denoted by $(\mathcal{F})_L$. Interestingly, every L-DFS can be constructed based on some DFS.

So far the axioms of L-DFSes are based on those of DFSes and are, therefore, not generally suited to express branching quantification adequately. In order to further restrict the set of plausible models Glöckner requires models to be compatible with so-called quantifier nesting i.e. the representation of polyadic quantifiers by a composition of simpler quantifiers.

Example 4 (Quantifier nestings [7])*.* Let $\mathcal{F}$ be an L-DFS and consider a semi-fuzzy L-quantifier of the form $Q'_x Q''_y \varphi(x, y)$. There are now two possibilities to analyze this quantifier in order to produce its fuzzy counterpart via $\mathcal{F}$.

First we might analyze the quantifier as a single type $\langle 2 \rangle$ semi-fuzzy L-quantifier $Q' \mathbin{\widetilde{@}} Q'' : \mathcal{P}(E^2) \to \mathbf{I}$ defined by

$$(Q' \mathbin{\widetilde{@}} Q'')(S) = \mathcal{F}(Q')(Z), \text{ for all } S \in \mathcal{P}(E^2),$$

where $Z \in \widetilde{\mathcal{P}}(E)$ is characterized by $\mu_Z(e) = Q''(eS)$ and $eS = \{e' \in E : (e, e') \in S\}$. We thus obtain the fuzzy L-quantifier $\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'') : \widetilde{\mathcal{P}}(E^2) \to \mathbf{I}$.

Alternatively we also could analyze the quantifier given above as a succession of quantifiers of type $\langle 1 \rangle$. Then we would obtain its fuzzy counterpart by translating Q', Q'' to $\mathcal{F}(Q')$ and $\mathcal{F}(Q'')$ and composing these fuzzy quantifiers following the same structure. This yields the fuzzy L-quantifier $(\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')) : \widetilde{\mathcal{P}}(E^2) \to \mathbf{I}$ given by

$$(\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q''))(R) = \mathcal{F}(Q')(Z), \text{ for all } R \in \widetilde{\mathcal{P}}(E^2),$$

where $Z \in \widetilde{\mathcal{P}}(E)$ is characterized by $\mu_Z(e) = \mathcal{F}(Q'')(eR)$ and $eR \in \widetilde{\mathcal{P}}(E)$ is characterized by $\mu_{eR}(e') = \mu_R(e, e')$.

We have thus obtained two possibly different models of the same quantifier by analyzing it in different ways. It therefore seems reasonable to require that these two models coincide in plausible L-DFSes i.e. that

$$\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'') = \mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'').$$

□

This restriction however turned out to be very constraining or even to be incompatible with the existing axioms. Glöckner points out that it would require further research in order to identify the L-DFSes best suited to model branching quantification.

5 Further Works

This section contains some useful advice for further study of the modeling of fuzzy quantifiers and gives the reader some directions for further research topics.

First of all the interested reader might want to deepen his knowledge of the state of the art in fuzzy quantifier modeling. To this end, the reader is advised to consult the overview established by Delgado et al. [5]. This work is a compilation of most of the approaches to fuzzy quantification. It contains short descriptions of the approaches and their respective adequacy properties, computational complexity, etc. Furthermore, the reader might be interested in more details about the particularly sophisticated QFM approach conceived by Glöckner. A detailed explanation about the DFS-Theory as well as elaborate analyses of a number of approaches can be found in Glöckner's book [7]. Besides this Glöckner's book also contains several discussions of the notions of vagueness and quantification and represents, therefore, a good reading to broaden the understanding of fuzzy quantification.

Throughout the overview we have seen several approaches that are all more or less systematically analyzable for adequacy. The particular case of the QFM approach provided by Glöckner is well understood from the perspective of its linguistic adequacy. This is not only due to the generality of the proposed approach, which allowed to consider whole classes of models at once, but also because it was provided with a framework that defines a set of adequacy constraints and concepts such as induced truth functions and extension principles. The framework, therefore, provides a solid context in which Glöckner's models are situated. Most, if not all, of the previous approaches lacked for such a framework and were thus much harder to situate in the context of a fuzzy system. It would, therefore, be reasonable to develop new methods within a similar framework. This is in particular the case of the approaches based on representation levels which have so far not be the subject of rigorous and profound analysis. Other interesting research topics are related to Glöckner's framework, which covers many aspects of quantification but which might be completed with respect to branching quantification and the quantification over masses, which would require a measure theoretic approach [7].

Conclusion

In this paper we gave an overview over the various approaches to the modeling of fuzzy quantifiers. To this end, we presented the formal and linguistic notions of vagueness, fuzziness, and quantification, which constitute the background of fuzzy quantification. Furthermore, we distinguished between two main approaches to fuzzy quantification: the fuzzy linguistic quantifier approach and the semi-fuzzy quantifier approach. Our examination of the simpler, but less general fuzzy linguistic quantifier methods was carried out from a slightly more practical point of view by providing examples demonstrating their properties and behavior. The more sophisticated semi-fuzzy quantifier methods were discussed from a more theoretical perspective and we paid special attention to the DFS theory developed by Glöckner. Finally, we gave a brief overview of the method developed by Glöckner for the modeling of fuzzy polyadic quantification.

After the reading of this article the reader should be familiar with the notions of vagueness, fuzziness and quantification. Moreover, the reader should have a basic understanding of the techniques applied in the modeling of fuzzy quantifiers, enabling him to find an easy access to the related literature.

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